

In this note, we aim to show that imposing the constraint $S^{i0} = 0$ in Schwarzschild spacetime leads to physically unusual results. In fact, it is crucial that in the analysis of spin–gravity coupling and the motion of a spinning particle in a gravitational field, an appropriate supplementary condition must be chosen in order to preserve physical consistency. We will demonstrate that the use of the $S^{i0} = 0$ condition leads to the prediction of an electromagnetic-like spectrum in the vicinity of a Schwarzschild black hole. The condition $S^{i0} = 0$ was historically employed in early literature on the motion of spinning particles in gravitational fields, and is notably present in the early work of Papapetrou. In the previous note, the general derivation of the MPD equations based on the $S^{i0} = 0$ condition was examined in detail.

Based on the equation for the ϕ -component of the MPD equations derived in the previous note, the ϕ -component equation reduces to the following expression for motion confined to a single plane:

$$\frac{d(M_{\text{eff}}\dot{\phi})}{d\tau} + \frac{2\dot{r}}{r} (M_{\text{eff}}\dot{\phi}) + \frac{3a'}{2r} \dot{r} S^{13} = 0 \quad (1)$$

$$M_{\text{eff}}\dot{\phi} := A \quad (2)$$

$$\frac{dA}{d\tau} + \frac{2\dot{r}}{r} A + \frac{3a'}{2r} \dot{r} S^{13} = 0 \quad (3)$$

From the results obtained in the previous note, we have:

$$S^{13} = \frac{S_0}{r} e^a \quad (4)$$

$$\Rightarrow \frac{dA}{dr} + \frac{2A}{r} + \frac{3a'}{2r} \frac{S_0}{r} e^a = 0 \quad (5)$$

$$A' + \frac{2A}{r} + \frac{3S_0}{2r^2} e^a a' = 0 \quad (6)$$

$$e^a a' = \frac{2GM}{r^2} \quad (7)$$

$$\Rightarrow A' + \frac{2A}{r} + \frac{3GMS_0}{r^4} = 0 \quad (8)$$

$$\Rightarrow r^4 A' + 2r^3 A + 3GMS_0 = 0 \quad (9)$$

Differentiating the above equation with respect to (r), we obtain:

$$\Rightarrow 4r^3 A' + r^4 A'' + 6r^2 A + 2r^3 A' = 0 \quad (10)$$

$$\Rightarrow r^4 A'' + 6r^3 A' + 6r^2 A = 0 \quad (11)$$

$$\Rightarrow r^2 A'' + 6r A' + 6A = 0 \quad (12)$$

The above equation is a Cauchy–Euler differential equation. Its solutions can therefore be determined as follows:

$$A = r^\lambda \quad (13)$$

$$\Rightarrow \lambda(\lambda - 1)r^\lambda + 6\lambda r^\lambda + 6r^\lambda = 0 \quad (14)$$

$$\Rightarrow \lambda(\lambda - 1) + 6\lambda + 6 = 0 \quad (15)$$

$$\lambda^2 + 5\lambda + 6 = 0 \quad (16)$$

$$\Rightarrow (\lambda + 3)(\lambda + 2) = 0 \quad (17)$$

$$\lambda = -2, -3 \quad (18)$$

The general solution to the differential equation is given by:

$$A = c_1 r^{-3} + c_2 r^{-2} \quad (19)$$

Substituting the above general solution into Eq.(9), we obtain:

$$\Rightarrow \begin{cases} A = c_1 r^{-3} : & -3c_1 + 2c_1 + 3GMS_0 = 0 \Rightarrow c_1 = 3GMS_0 \\ A = c_2 r^{-2} : & -2c_2 + 2c_2 + 3GMS_0 = 0 \Rightarrow c_2 = 0 \end{cases} \quad (20)$$

$$\Rightarrow A = \frac{3GMS_0}{r^3} \quad (21)$$

$$\boxed{M_{\text{eff}}\dot{\phi} = \frac{3GMS_0}{r^3}} \quad (22)$$

Now, taking into account the calculations of the previous note and the assumption made in this section that the motion is constrained to a plane, we obtain:

$$m_s = \frac{a'}{2} \left[g_{22} \dot{\theta} S^{12} + g_{33} \dot{\phi} S^{13} \right] \quad (23)$$

$$\dot{\theta} = 0 \quad (24)$$

$$m_s = \frac{a'}{2} (-r^2) \dot{\phi} \frac{S_0}{r} e^a = \left(-\frac{r \dot{\phi} a' e^a}{2} S_0 \right) \quad (25)$$

$$m_s = -\frac{r \dot{\phi}}{2} S_0 \frac{2GM}{r^2} = -\frac{GMS_0}{r} \dot{\phi} \quad (26)$$

By substituting the above relation into Eq.(22), we obtain:

$$\left(m - \frac{GMS_0}{r} \dot{\phi} \right) \dot{\phi} = \frac{3GMS_0}{r^3} \quad (27)$$

$$\frac{GMS_0}{r} \dot{\phi}^2 - m \dot{\phi} + \frac{3GMS_0}{r^3} = 0 \quad (28)$$

$$\dot{\phi} = \frac{m \pm \sqrt{m^2 - \frac{12}{r^4} (GMS_0)^2}}{\frac{2GMS_0}{r}} \quad (29)$$

Again, using Eq. (22), we have:

$$M_{\text{eff}} = \frac{m \mp \sqrt{m^2 - 12 \left(\frac{GMS_0}{r^2} \right)^2}}{2} \quad (30)$$

Taking into account that as $r \rightarrow \infty$, the effective mass must approach (m) (since the gravitational field becomes negligible, the spacetime becomes flat, and consequently the spin-curvature coupling vanishes), we obtain:

$$M_{\text{eff}} = m \left[\frac{1}{2} + \frac{\sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2}}{2} \right] \quad (31)$$

$$\dot{\phi} = \frac{mr}{2GMS_0} \left[1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2} \right] \quad (32)$$

Clearly, for a particle moving in a plane, we must be able to define $\dot{\phi}$. In order for this quantity to be real-valued, the expression under the square root must be non-negative. That is:

$$1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2 \geq 0 \quad (33)$$

$$\left(\frac{GMS_0}{mr^2} \right)^2 \leq \frac{1}{12} \quad (34)$$

$$-\frac{1}{2\sqrt{3}} \leq \frac{GMS_0}{mr^2} \leq \frac{1}{2\sqrt{3}} \quad (35)$$

$$\Rightarrow \begin{cases} -mr^2 \leq 2\sqrt{3} GMS_0 \\ 2\sqrt{3} GMS_0 \leq mr^2 \end{cases} \quad (36)$$

$$\Rightarrow \begin{cases} r^2 \geq -\frac{2\sqrt{3} GMS_0}{m} \\ r^2 \geq \frac{2\sqrt{3} GMS_0}{m} \end{cases} \quad (37)$$

$$r \geq 0 \Rightarrow \boxed{r \geq \sqrt{\frac{2\sqrt{3} GM|S_0|}{m}}} \quad (38)$$

This result is particularly important because it imposes a constraint on the particle's orbit, implying that for particles with different spin and mass, the lower bound of their distance from the black hole's center will differ. In general, one must carefully consider the regime in which the calculation is performed. Given the approximations used, applying these equations to a quantum particle should be treated with caution, and the formalism is essentially better interpreted as a classical or semiclassical description.

For this reason, extending the above equations and results to photons is also questionable (Although in several early papers on the motion of spinning particles in gravitational fields, the results were explicitly extended to photons.). However, let us, for the sake of argument, assume that such an extension is meaningful. By replacing m in relation (39) with the photon energy divided by c^2 , and recalling that in quantum mechanics the photon energy is given by $E = hf$, where f is the frequency of light, we would then find that photons exhibit a form of spectral structure around the black hole.

In this picture, one obtains a kind of "spectral mapping" around the Schwarzschild black hole: depending on the frequency, the allowed regions of propagation form nested concentric circular shells. Certain frequencies would be forbidden in regions where other frequencies are allowed, leading to a spatially dependent distribution of photon density and intensity. In principle, one could compute the photon density in different regions via a straightforward calculation.

However, the physical validity of such an interpretation is highly questionable. Several issues arise in this context: first, the application of the above results to photons is not justified given the approximations and the essentially classical nature of the underlying equations. More importantly, the supplementary condition used earlier does not necessarily carry a clear physical meaning in this regime.

As discussed previously, different supplementary conditions must be carefully examined to determine which ones are physically consistent. The condition used here can be interpreted as a special case of the general orthogonality condition between the spin tensor and the four-velocity. In fact, it corresponds to working in the center-of-mass frame of the system, where such an orthogonality condition is imposed. From this perspective, the observed structure might be a consequence of this specific frame choice.

Alternatively, one may treat this condition as an independent constraint rather than a special case of the general orthogonality condition. In that case, the situation becomes even more puzzling, since one would then expect this spectral structure to be, in principle, observable—despite the significant experimental challenges involved.

Overall, our main conclusion is that, in the Schwarzschild black hole context, imposing this supplementary condition does not lead to a physically consistent interpretation of nature. Instead, it appears to be primarily a mathematical construction—perhaps meaningful in an abstract or hypothetical framework, but not in realistic physical scenarios.

Now let us determine the range of the mass as well. Given the range obtained for (r) , we can similarly derive corresponding bounds for (m_s) and the effective mass:

$$m_s = (-GMS_0) \frac{m}{2GMS_0} \left[1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2} \right] \quad (39)$$

$$m_s = -m \left[\frac{1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2}}{2} \right] \quad (40)$$

$$\Rightarrow \boxed{-\frac{m}{2} \leq m_s \leq 0} \quad (41)$$

$$\Rightarrow \boxed{\frac{m}{2} \leq M_{\text{eff}} \leq m} \quad (42)$$

Now, based on the calculations of the previous note, we write Q^1 for motion constrained to a plane:

$$Q^1 = \frac{a'}{2} e^{2a} \dot{t}^2 - \frac{a'}{2} \dot{r}^2 - r e^a \dot{\phi}^2 \quad (43)$$

Now, substituting the expressions for $\dot{\phi}$ and the effective mass from Eqs. (32) and (33), and performing some simplification, we obtain:

$$Q^1 = \frac{a'}{2} \frac{E^2}{M_{\text{eff}}^2} - \frac{a'}{2} \dot{r}^2 - r e^a \left[\frac{mr\dot{\phi}}{GMS_0} - \frac{3}{r^2} \right] \quad (44)$$

$$Q^1 = \frac{a' r^6 E^2}{18(GMS_0)^2} \left[\frac{mr\dot{\phi}}{GMS_0} - \frac{3}{r^2} \right] - \frac{a'}{2} \dot{r}^2 + \frac{3e^a}{r} - \frac{mr^2 e^a \dot{\phi}}{GMS_0} \quad (45)$$

$$Q^1 = \frac{mr^2\dot{\phi}}{GMS_0} \left[\frac{a'r^5E^2}{18(GMS_0)^2} - e^a \right] + \frac{3e^a}{r} - \frac{a'\dot{r}^2}{2} - \frac{a'r^4E^2}{6(GMS_0)^2} \quad (46)$$

$$Q^1 = r \left(\frac{mr}{GMS_0} \right)^2 \left[\frac{a'r^5E^2}{18(GMS_0)^2} - e^a \right] \left[\frac{1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2}}{2} \right] + \frac{3e^a}{r} - \frac{a'\dot{r}^2}{2} - \frac{a'r^4E^2}{6(GMS_0)^2} \quad (47)$$

$$Q^1 = \frac{m^2r^3}{2(GMS_0)^2} \left[\frac{a'r^5E^2}{18(GMS_0)^2} - e^a \right] \left[1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2} \right] + \frac{3e^a}{r} - \frac{a'\dot{r}^2}{2} - \frac{a'r^4E^2}{6(GMS_0)^2} \quad (48)$$

Similarly, ϕ' , which is generally more physically useful for analyzing the particle's trajectory, can be written as:

$$\phi' = \frac{mr}{2GMS_0\dot{r}} \left[1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2} \right] \quad (49)$$

We now compute $\frac{1}{\dot{r}}Q^1$:

$$\frac{1}{\dot{r}}Q^1 = \left[\frac{m^2r^3}{2(GMS_0)^2} \left(\frac{a'r^5E^2}{18(GMS_0)^2} - e^a \right) \left(1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2} \right) + \frac{3e^a}{r} - \frac{a'r^4E^2}{6(GMS_0)^2} \right] \frac{1}{\dot{r}} - \frac{a'\dot{r}}{2} \quad (50)$$

We define (W) , which is solely a function of (r) , as follows:

$$W_{(r)} := \frac{m^2r^3}{2(GMS_0)^2} \left(\frac{a'r^5E^2}{18(GMS_0)^2} - e^a \right) \left(1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2} \right) + \frac{3e^a}{r} - \frac{a'r^4E^2}{6(GMS_0)^2} \quad (51)$$

From Eq. (50), \dot{r} is given by:

$$\dot{r} = \frac{mr}{GMS_0\phi'} \left[\frac{1 - \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2}}{2} \right] \quad (52)$$

We now rewrite the radial MPD equation obtained in the previous note in the following form:

$$\dot{r} \frac{dV}{dr} + \frac{Q^1}{\dot{r}} V = 0 \quad (53)$$

where (V) is defined as follows:

$$\dot{r} = \frac{V}{M_{\text{eff}}} \quad (54)$$

With some simplification and using Eqs. (52) and (53), we obtain:

$$\frac{V}{M_{\text{eff}}} \frac{dV}{dr} + \left(\frac{W_{(r)}}{\dot{r}} - \frac{a'\dot{r}}{2} \right) V = 0 \quad (55)$$

$$\frac{V}{M_{\text{eff}}} \frac{dV}{dr} + \left(\frac{W_{(r)}}{V} M_{\text{eff}} - \frac{a'}{2} \frac{V}{M_{\text{eff}}} \right) V = 0 \quad (56)$$

$$\frac{V}{M_{\text{eff}}} \frac{dV}{dr} + M_{\text{eff}} W_{(r)} - \frac{a'}{2M_{\text{eff}}} V^2 = 0 \quad (57)$$

$$\boxed{VV' + M_{\text{eff}}^2 W_{(r)} - \frac{a'}{2} V^2 = 0} \quad (58)$$

We define:

$$V^2 := X \quad \Rightarrow \quad 2VV' = X' \quad (59)$$

By substituting, we obtain:

$$\frac{X'}{2} - \frac{a'}{2} X + M_{\text{eff}}^2 W_{(r)} = 0 \quad (60)$$

$$\boxed{X' - a'X + 2M_{\text{eff}}^2 W_{(r)} = 0} \quad (61)$$

We now proceed to solve the above differential equation:

$$dX + (-a'X + 2M_{\text{eff}}^2 W(r)) dr = 0 \quad (62)$$

We need to determine the integrating factor $\mu(r)$. For this purpose, from the theory of differential equations, we know that the following relation must hold:

$$\frac{\partial}{\partial r}(\mu) = \frac{\partial}{\partial X}(-\mu a'X + 2\mu M_{\text{eff}}^2 W(r)) \quad (63)$$

$$\mu' = -\mu a' \quad (64)$$

$$\frac{d\mu}{\mu} = -da \quad (65)$$

$$\boxed{\mu = e^{-a} = \frac{1}{1 - \frac{2GM}{r}}} \quad (66)$$

We now multiply by the integrating factor:

$$\frac{dX}{1 - \frac{2GM}{r}} + \left(\frac{-a'X}{1 - \frac{2GM}{r}} + \frac{2M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} \right) dr = 0 \quad (67)$$

$$\frac{dX}{1 - \frac{2GM}{r}} + \left(\frac{-\frac{2GM}{r^2}}{\left(1 - \frac{2GM}{r}\right)^2} X + \frac{2M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} \right) dr = 0 \quad (68)$$

Assume that the following equation holds:

$$\psi_{(X,r)} = 0 \Rightarrow d\psi = 0 \Rightarrow \frac{\partial\psi_{(X,r)}}{\partial X} dX + \frac{\partial\psi_{(X,r)}}{\partial r} dr = 0 \quad (69)$$

By equating the coefficients of (dX) and (dr) in Eqs. (68) and (69), we obtain:

$$\begin{cases} \frac{\partial\psi_{(X,r)}}{\partial X} = \frac{1}{1 - \frac{2GM}{r}} \\ \frac{\partial\psi_{(X,r)}}{\partial r} = -\frac{\frac{2GM}{r^2}}{\left(1 - \frac{2GM}{r}\right)^2} X + \frac{2M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} \end{cases} \quad (70)$$

We now simplify the second equation of Eq. (71) by integrating with respect to (r):

$$\frac{\partial\psi}{\partial r} = -\frac{2GM}{(r - 2GM)^2} X + \frac{2M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} \quad (71)$$

$$\psi = -2GMX \int \frac{dr}{(r - 2GM)^2} + 2 \int \frac{M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} dr + \Phi(X) \quad (72)$$

$$\psi = \frac{2GM}{r - 2GM} X + 2 \int \frac{M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} dr + \bar{\Phi}(X) \quad (73)$$

where Φ is a function that depends only on (X) and is independent of (r). From the first equation of Eq. (71), we have:

$$\frac{\partial\psi}{\partial X} = \frac{1}{1 - \frac{2GM}{r}} \quad (74)$$

Now, from Eqs. (74) and (75), we obtain:

$$\frac{2GM}{r - 2GM} + \frac{\partial\bar{\Phi}}{\partial X} = \frac{1}{1 - \frac{2GM}{r}} \quad (75)$$

$$\frac{\partial\bar{\Phi}}{\partial X} = \frac{r - 2GM}{r - 2GM} = 1 \quad (76)$$

$$\bar{\Phi} = X + \text{constant} := X + \bar{\Phi}_0 \quad \bar{\Phi}_0 : \text{constant} \quad (77)$$

By substituting the above result into Eq. (74), we obtain:

$$\psi = \left(\frac{2GM}{r-2GM} + 1 \right) X + 2 \int \frac{M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} dr + \bar{\Phi}_0 \quad (78)$$

According to Eq. (70), the above expression is equal to zero:

$$\psi = \frac{X}{1 - \frac{2GM}{r}} + 2 \int \frac{M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} dr + \bar{\Phi}_0 = 0 \quad (79)$$

$$\boxed{X = - \left(1 - \frac{2GM}{r} \right) \left[2 \int \frac{M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} dr + \bar{\Phi}_0 \right]} \quad (80)$$

We now explicitly evaluate the integral appearing in Eq. (81):

$$M_{\text{eff}}^2 = \frac{m^2}{4} (1 + \nu)^2, \quad \nu := \sqrt{1 - 12 \left(\frac{GMS_0}{mr^2} \right)^2} \quad (81)$$

$$I = \int \frac{M_{\text{eff}}^2 W(r)}{1 - \frac{2GM}{r}} dr \quad (82)$$

$$e^{-a} W(r) M_{\text{eff}}^2 = \frac{m^2 r^3}{2(GMS_0)^2} \left[\frac{e^{-a} a' r^5 E^2}{18(GMS_0)^2} - 1 \right] [1 - \nu] \frac{(1 + \nu)^2 m^2}{4} + \frac{3m^2}{r} \frac{1}{4} (1 + \nu)^2 - \frac{e^{-a} a' r^4 E^2}{6(GMS_0)^2} \frac{m^2}{4} (1 + \nu)^2 \quad (83)$$

$$= \frac{3m^2}{2r} \left[\frac{e^{-a} a' r^5 E^2}{18(GMS_0)^2} - 1 \right] (1 + \nu) + \frac{3m^2}{4r} (1 + 2\nu + \nu^2) - \frac{e^{-a} a' r^4 E^2}{6(GMS_0)^2} \frac{m^2}{4} (1 + 2\nu + \nu^2) \quad (84)$$

$$= \frac{e^{-a} a' r^4 E^2 m^2}{12(GMS_0)^2} \left(\frac{1}{2} - \frac{\nu^2}{2} \right) + \frac{3m^2}{2r} \left(-\frac{1}{2} + \frac{\nu^2}{2} \right) \quad (85)$$

$$= \frac{1 - \nu^2}{2} \left[\frac{e^{-a} a' r^4 E^2 m^2}{12(GMS_0)^2} - \frac{3m^2}{2r} \right] \quad (86)$$

$$= 6 \left(\frac{GMS_0}{mr^2} \right)^2 \frac{m^2}{2} \left[\frac{e^{-a} a' r^4 E^2}{6(GMS_0)^2} - \frac{3}{r} \right] \quad (87)$$

$$= 3 \left(\frac{GMS_0}{r^2} \right)^2 \left[\frac{2GM r^4 E^2}{6(GMS_0)^2 (r - 2GM)^2} - \frac{3}{r} \right] \quad (88)$$

$$= \frac{GME^2}{(r - 2GM)^2} - \frac{9(GMS_0)^2}{r^5} \quad (89)$$

The above simplifications lead to the following result:

$$I = GME^2 \int \frac{dr}{(r - 2GM)^2} - 9(GMS_0)^2 \int \frac{dr}{r^5} \quad (90)$$

$$= -\frac{GME^2}{r - 2GM} - 9(GMS_0)^2 \left(-\frac{1}{4r^4} \right) \quad (91)$$

$$= -\frac{GME^2}{r - 2GM} + \frac{9(GMS_0)^2}{4r^4} \quad (92)$$

$$\Rightarrow \boxed{2I = \frac{E^2}{1 - \frac{r}{2GM}} + \frac{9(GMS_0)^2}{2r^4}} \quad (93)$$

Using the above result together with Eq. (81), we obtain:

$$X = \left(-1 + \frac{2GM}{r} \right) \left[\frac{E^2}{1 - \frac{r}{2GM}} + \frac{9(GMS_0)^2}{2r^4} + \bar{\Phi}_0 \right] \quad (94)$$

$$\boxed{X = \frac{2GME^2}{r} - \frac{9(GMS_0)^2}{2} \left(\frac{1 - \frac{2GM}{r}}{r^4} \right) - \left(1 - \frac{2GM}{r} \right) \bar{\Phi}_0} \quad (95)$$

$$X = (M_{\text{eff}} \dot{r})^2 \quad (96)$$

$$\frac{X}{(M_{\text{eff}} \dot{\phi})^2} = \left(\frac{dr}{d\phi} \right)^2 \quad (97)$$

$$\Rightarrow \boxed{\frac{dr}{d\phi} = \pm \sqrt{-\frac{\bar{\Phi}_0}{9(GMS_0)^2} r^6 + \frac{2GM(E^2 + \bar{\Phi}_0)}{9(GMS_0)^2} r^5 - \frac{r^2}{2} + GMr}} \quad (98)$$

This equation, together with the initial conditions, determines the trajectory of the particle. The only remaining step is to express S_0 in terms of the conserved quantities.

In the previous note, we showed that the angular momentum components along the x, y, and z directions are conserved. We now aim to demonstrate that the angular momentum in the z-direction is equal to S_0 . For this purpose, we employ the θ -component of the MPD equations derived in the previous note for motion constrained to a plane:

$$\frac{1}{r} \dot{r} S^{12} + r e^a \dot{\phi} S^{23} = 0 \quad (99)$$

According to the spin evolution equation derived in the previous note, we have:

$$\frac{dS^{12}}{d\tau} + \left(\frac{1}{r} - a'\right) \dot{r} S^{12} + r e^a \dot{\phi} S^{23} = 0 \quad (100)$$

$$\frac{dS^{12}}{d\tau} + \left(\frac{1}{r} - a'\right) \dot{r} S^{12} + r e^a \left[-\frac{\dot{r} S^{12}}{r^2 e^a}\right] = 0 \quad (101)$$

$$\frac{dS^{12}}{S^{12}} = da \quad (102)$$

$$d(\ln |S^{12}| - a) = 0 \quad (103)$$

$$\Rightarrow \boxed{S^{12} = \tilde{S}_0 e^a} \quad \tilde{S}_0 : \text{constant} \quad (104)$$

Using the relations obtained in the previous note, we have:

$$S^{23} = -\frac{S^{12}}{r^2 e^a} \frac{dr}{d\phi} = -\frac{\tilde{S}_0}{r^2} \frac{dr}{d\phi} \quad (105)$$

By substituting from Eq. (99), we obtain:

$$\boxed{S^{23} = \pm \tilde{S}_0 \sqrt{-\frac{\bar{\Phi}_0 r^2}{9(GMS_0)^2} + \frac{2GM(E^2 + \bar{\Phi}_0)}{9(GMS_0)^2} r - \frac{1}{2r^2} + \frac{GM}{r^3}}} \quad (106)$$

Using the results obtained above and the expression for the z-component of angular momentum derived in the previous note, we have:

$$L_z = r^2 \left(\frac{3GMS_0}{r^3} - \frac{a'}{2} \frac{S_0 e^a}{r} \right) + e^a S_0 \quad (107)$$

$$L_z = \frac{3GMS_0}{r} - \frac{a'r}{2} S_0 e^a + e^a S_0 \quad (108)$$

$$L_z = \frac{3GMS_0}{r} + S_0 e^a \left(1 - \frac{a'r}{2} \right) \quad (109)$$

$$e^a a' = \frac{2GM}{r^2} \quad (110)$$

$$\frac{a'r}{2} = \frac{GM}{r e^a} = \frac{\frac{GM}{r}}{1 - \frac{2GM}{r}} \quad (111)$$

$$1 - \frac{a'r}{2} = \frac{1 - \frac{3GM}{r}}{e^a} \quad (112)$$

$$\Rightarrow L_z = \frac{3GMS_0}{r} + S_0 \left(1 - \frac{3GM}{r} \right) \quad (113)$$

$$\boxed{L_z = S_0} \quad (114)$$

Here, we have determined all constants in terms of the key physical quantities of the problem, and in this way we have been able to explicitly compute the radial and angular “velocities” of the particle. This implies that, given appropriate initial conditions, the motion of the particle can be fully predicted (Eq. 99).